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THE STABILITY OF A PLATE OF SHAPE-MEMORY ALLOY IN A DIRECT THERMOELASTIC PHASE TRANSITION†

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Analytical solutions are obtained in various formulations of the coupled problem of stability for a rectangular plate of a shape-memory alloy undergoing a direct thermoelastic phase transition when compressive loads are applied. It is shown that the critical loads corresponding to the coupled formulation of the problem may be several times less than those obtained when solving the uncoupled problem. A non-convex domain of stability is obtained in the plane of the applied loads. © 2004 Elsevier Ltd. All rights reserved.

Shape-memory alloys (SMAs) possess unique mechanical properties because of the thermoelastic phase transitions that take place in them. When such an alloy is cooled, a direct phase transition occurs in it in an appropriate temperature interval: from the austenite phase to the martensite phase, accompanied by a decrease in Young's modulus (down to as much as one third in titanium nickelide). If the direct transition takes place under mechanical stresses, the SMA will also experience, besides an elastic strain, a phase strain exceeding the elastic strain by a considerable factor, corresponding to the same stresses. The decrease in elastic stiffness and increase in the deformative property of the SMA in a direct transition indicates the danger of a loss of stability when the phase transition occurs as a result of the application of compressive stresses.

The stability of the equilibrium of the elements of a SMA was investigated in [1–6]. Experiments have established [5] that samples in the form of thin strips of titanium nickelide, which lose isothermal stability neither in the austenite nor in the martensite state, may lose stability under the same load in a transition from the first state to the second due to cooling. The critical stability-loss loads in a direct martensite transition proved to be several times less than the critical loads for loss of isothermal stability in the least rigid martensite phase state, so that the observed phenomenon cannot be attributed merely to a decrease in the moduli of elasticity under a direct martensite transition.

For a qualitative description of the experimentally observed phenomenon, analytical solutions of the stability problem are obtained in this paper for a rectangular SMA plate undergoing direct martensitic transition under unilateral and bilateral uniform loading. The problem is solved in different formulations, for the purpose of choosing a solution that yields the lowest critical loads.

1. FORMULATION OF THE PROBLEM

Consider a rectangular plate of constant thickness h and sides a and b along the axes of a Cartesian system of coordinates Ox_1 and Ox_2 situated in the middle plane of the plate (the origin O of the system of coordinates coincides with one of the nodes of the plate). The plate is loaded in the austenite phase state by constant normal surface forces p_{11} and p_{22}, acting in the directions of the axes Ox_1 and Ox_2, respectively (a compressive load is considered positive), uniformly distributed over opposite edges; the plate is cooled slowly from the start temperature to the finish temperature of the direct martensitic transition. At any given instant of time, all points of the plate are at the same temperature. It is required to find the minimum loads at which the plate may have twisted equilibrium shapes during the direct transition process, in addition to its trivial flat shape. The investigation will be carried out within the limits of small-strain theory and the Kirchhoff–Love hypothesis (for total strains). The problem of stability will be solved in a linearized formulation.

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A simplified version of the system of constitutive relations proposed in [7–9] for shape-memory alloys (SMAs) will be used. For the case of direct transition in a plane stressed state, this system reduces to the following

$$\varepsilon_{ij} = \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)} \tag{1.1}$$

$$d\varepsilon_{11}^{(2)} = \left(\frac{2\sigma_{11} - \sigma_{22}}{3\sigma_{(1)}} + a_0\varepsilon_{11}^{(2)}\right)dq, \quad d\varepsilon_{22}^{(2)} = \left(\frac{2\sigma_{22} - \sigma_{11}}{3\sigma_{(1)}} + a_0\varepsilon_{22}^{(2)}\right)dq$$
$$d\varepsilon_{12}^{(2)} = \left(\frac{\sigma_{12}}{\sigma_{(1)}} + a_0\varepsilon_{12}^{(2)}\right)dq \tag{1.2}$$

$$q = \sin\left(\frac{\pi}{2}\frac{M_1 + k\sigma_i - T}{M_1 - M_2}\right) \quad \text{or} \quad q = \frac{1}{2}\left[1 - \cos\left(\pi\frac{M_1 + k\sigma_i - T}{M_1 - M_2}\right)\right]$$
$$M_2 + k\sigma_i \le T \le M_1 + k\sigma_i, \quad kd\sigma_i - dT > 0 \tag{1.3}$$

$$\sigma_i = \sqrt{\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 3\sigma_{12}^2} \tag{1.4}$$

$$\varepsilon_{11}^{(1)} = \frac{\sigma_{11} - \mu(q)\sigma_{22}}{E(q)}, \quad \varepsilon_{22}^{(1)} = \frac{\sigma_{22} - \mu(q)\sigma_{11}}{E(q)}, \quad \varepsilon_{12}^{(1)} = \frac{\sigma_{12}}{2G(q)} \tag{1.5}$$

$$\frac{1}{E(q)} = \frac{q}{E_1} + \frac{1-q}{E_2}, \quad \frac{1}{G(q)} = \frac{q}{G_1} + \frac{1-q}{G_2}, \quad \mu(q) = \frac{E(q)}{2G(q)} - 1 \tag{1.6}$$

where ε_{ij}, $\varepsilon_{ij}^{(1)}$ and $\varepsilon_{ij}^{(2)}$ are the total, elastic and phase strains, σ_{ij} and σ_i are the stress tensor and intensity, q is the internal state variable, treated as the relative volume of the martensite phase, for which the first [9] or second [10] relation of (1.3) is used, T is the variable temperature, and M_1 and M_2 are the start and finish of the direct martensitic transition in the stress-free material. The third and fourth relations in (1.3) are the conditions for the realization of a direct martensitic transition, $E(q)$; $G(q)$ and $\mu(q)$ are the Young's modulus, the shear modulus and Poisson's ratio of the SMA, whose dependence on the phase composition parameter is defined by formulae (1.6), which are obtained by hypothesizing additivity of the Gibb's potential and Reuss averaging [11] the subscripts 1 and 2 indicate the values of the moduli for the martensite and austenite states, respectively), and a_0, $\sigma_{(1)}$ and k are the material constant for the SMA.

According to Eqs (1.1) and (1.2), neither the volume effect of the martensite transition reaction nor pure temperature strains are taken into consideration when solving the problem; it is moreover obvious from Eqs (1.3) and (1.4) that the influence of transverse shear stresses is ignored when calculating of the start and finish temperatures of the direct transition.

Suppose we can set $k = 0$ in Eqs (1.3). The boundary-value problems arising on the basis of the resulting system may be classified, following the terminology adopted in [11], as unconnected, since in that case the distribution of the phase composition parameter over the material may be found independently of solving the problem of determining the stress-strain state. On the assumption that the transition to the adjacent equilibrium shape takes place much faster than the cooling process, and the falling temperature does not experience perturbations ($\delta T = 0$), one can arrive at a "fixed phase composition" conception [6], following which it may be assumed that, when a transition to the adjacent equilibrium shape occurs, the phase composition does not change ($\delta q = 0$). In the unconnected formulation, system (1.1)–(1.6) is used to analyse the unperturbed state, but in order to formulate the linearized equations of stability one uses, instead of Eqs (1.2) and (1.3), the relations

$$\delta q = 0, \quad \delta\varepsilon_{ij}^{(2)} = 0 \tag{1.7}$$

The case $k \ne 0$ corresponds to the connected statement of the stability problem. Here, even if $\delta T = 0$, transition to the adjacent equilibrium shape owing to variation of the stresses may be accompanied by additional phase transitions and the parameter q must be varied when the equations for the perturbed state are written down ("continuing phase transition" conception [6]).

The condition for realization of a phase transition according to the last relation of (1.3) is the validity of the inequality

$$k\delta\sigma_i - \delta T > 0 \tag{1.8}$$

In the case of buckling under a constant external load which experiences no perturbations, the compressive stresses may decrease near the convex surface of the plate and the stress intensity may decrease. If at the same time the temperature also does not experience any perturbations ($\delta T = 0$), condition (1.8) is violated. As a result this (hitherto unknown) part of the plate will not experience an additional phase transition when buckling occurs (the "elastic unloading" conception). Analysing the perturbed state for this part of the plate, one will use, as in the unconnected setting, relations (1.7), rather than Eqs (1.2) and (1.3), while for the other points of the cross-section both the preliminary stressed state and the transition to the adjacent equilibrium shape will be analysed using the complete system (1.1)–(1.6).

Suppose the acting load and/or temperature may experience small perturbations. It has been shown [6] that in the case of a rod of SMA there will always be infinitesimal variations of the load under which buckling will be accompanied by an increase in the stress intensity at all points of the cross-section. As a result, the whole cross-section will undergo an additional phase transition. The hypothesis that such perturbations exist and are realized corresponds to Shanley's conception in the theory of the stability of elastoplastic bodies [12, 13]; it may therefore be termed briefly the "continuing load" conception.

In experiments [5], the critical stability-loss loads in a direct martensite transition turned out to be extraordinarily low, and therefore it is interesting to find a formulation of the stability problem that will yield a decrease in the value of the critical loads.

2. LINEARIZED STABILITY EQUATIONS

To solve the stability problem one first has to find the unperturbed stress-strain and phase states. Following the general positions that have been established for the boundary-value problems of direct transition in SMAs [14, 15], one can show that the unperturbed stresses in a direct transition conserve the constant values

$$\sigma_{11} = -p_{11}, \quad \sigma_{22} = -p_{22}, \quad \sigma_{12} = 0 \tag{2.1}$$

In that case, integrating Eqs (1.2) taking the zero initial data into account, one can find the values of the phase strains in the unperturbed state

$$\varepsilon_{ii}^{(2)}(q) = \frac{2\sigma_{ii} - \sigma_{jj}}{3\sigma_{(1)}a_0}(\exp(a_0 q) - 1), \quad \varepsilon_{12}^{(2)}(q) = 0 \tag{2.2}$$

Here and below, there is no summation over repeated subscripts $i, j = 1, 2, i \neq j$.

The unperturbed values of q are given by formulae (1.3), (1.4) and (2.1) and will be the same at each instant of time for all points of the plate.

The equations for the perturbed state are obtained from the kinematic part of the Kirchhoff–Love hypotheses, which by virtue of (1.1) and (1.5), may be written as

$$\varepsilon_{ii} = \varepsilon_{ii}^0 - x_3\kappa_{ii} = \frac{\sigma_{ii} - \mu(q)\sigma_{jj}}{E(q)} + \varepsilon_{ii}^{(2)}, \quad \varepsilon_{12} = \varepsilon_{12}^0 - x_3\kappa_{12} = \frac{\sigma_{12}}{2G(q)} + \varepsilon_{12}^{(2)} \tag{2.3}$$

where x_3 is the coordinate orthogonal to the plate, ε_{ij}^0 is the strain of the middle plane of the plate, and κ_{ij} are the curvatures, for which the following linear relations are used

$$\kappa_{11} = w_{,11}, \quad \kappa_{22} = w_{,22}, \quad \kappa_{12} = w_{,12} \tag{2.4}$$

where w is the deflection of the middle plane of the plate, and the symbols following the comma in the subscript indicate the coordinates with respect to which the partial derivative is evaluated. In the discussion in this section, the whole set of variables mentioned previously are allowed to vary: w, q, p_{11} p_{22} and T.

Taking variations on both sides of Eqs (2.3), and considering the unperturbed values of the phase strains (2.2) and formulae (1.6), we obtain

$$\delta\varepsilon_{ii}^0 - x_3\delta\kappa_{ii} = \frac{\delta\sigma_{ii} - \mu(q)\delta\sigma_{jj}}{E(q)} + g_{ii}(q)\delta q, \quad \delta\varepsilon_{12}^0 - x_3\delta\kappa_{12} = \frac{\delta\sigma_{12}}{2G(q)}$$

$$g_{ii}(q) = \sigma_{ii}\Delta_1 - \sigma_{jj}\Delta_2 + \frac{2\sigma_{ii} - \sigma_{jj}}{3\sigma_{(1)}}\exp(a_0 q) \tag{2.5}$$

$$\Delta_1 = \left(\frac{1}{E_1} - \frac{1}{E_2}\right), \quad \Delta_2 = \left(\frac{\mu_1}{E_1} - \frac{\mu_2}{E_2}\right)$$

Here and below, all quantities not marked as variations correspond to the unperturbed state, and those with the variation symbol correspond to their variation on changing to the adjacent equilibrium shape.

For a variation of the relative volume of the martensite phase q in formulae (1.3), one can obtain the relation

$$\delta q = \psi(q)(k^*\delta\sigma_i + \delta t)U_+$$

$$t = \frac{M_1 - T}{M_1 - M_2}, \quad k^* = \frac{k}{M_1 - M_2}, \quad U_+ = \begin{cases} 1 & \text{if} \quad k^*\delta\sigma_i + \delta t > 0 \\ 0 & \text{if} \quad k^*\delta\sigma_i + \delta t \le 0 \end{cases} \tag{2.6}$$

If the first formula of (1.3) is being used, the quantity $\psi(q)$ is evaluated by the formula $\psi(q) = \pi\sqrt{1 - q^2}/2$; in the case of the second it is evaluated by the formula $\psi(q) = \pi\sqrt{q}(1 - q)$.

The variation of the stress intensity (1.4) (taking the absence of shear stresses in unperturbed compression into consideration) is computed by the formula

$$\delta\sigma_i = \frac{1}{\sigma_i}\left[\left(\sigma_{11} - \frac{\sigma_{22}}{2}\right)\delta\sigma_{11} + \left(\sigma_{22} - \frac{\sigma_{11}}{2}\right)\delta\sigma_{22}\right] \tag{2.7}$$

Solving Eq. (2.5) for the stress variations, we obtain

$$\delta\sigma_{ii} = \frac{E(q)}{1 - \mu^2(q)}[e_{ii}z_{jj} + e_{jj}z_{ij}], \quad \delta\sigma_{12} = 2G(q)(\delta\varepsilon_{12}^0 - x_3\delta\kappa_{12})$$

$$e_{ii} = \delta\varepsilon_{ii}^0 - x_3\delta\kappa_{ii} - g_{ii}(q)\psi(q)U_+\delta t$$

$$z_{jj} = \frac{1 + \xi_{jj}(q)U_+}{1 + \xi(q)U_+}, \quad z_{ij} = \frac{\mu(q) - \xi_{ij}(q)U_+}{1 + \xi(q)U_+}$$

$$\xi_{ii}(q) = \frac{g_{ii}(q)E(q)\psi(q)k^*}{\sigma_i}\left(\sigma_{ii} - \frac{\sigma_{jj}}{2}\right) \tag{2.8}$$

$$\xi_{ij}(q) = \frac{g_{ii}(q)E(q)\psi(q)k^*}{\sigma_i}\left(\sigma_{jj} - \frac{\sigma_{ii}}{2}\right)$$

$$\xi(q) = \frac{\xi_{11}(q) + \xi_{22}(q) + \mu(q)(\xi_{12}(q) + \xi_{21}(q))}{1 - \mu^2(q)}$$

Equations (2.8) hold for those points of the plate at which, on changing to the adjacent equilibrium shape, an additional phase transition occurs. For points where $\delta q = 0$, one should put $\psi(q) = 0$ in (2.8).

Using the stress variations (2.8), one can form the variations of the internal moments and forces per unit length.

$$\delta M_{ij} = \int_{-h/2}^{h/2} \delta\sigma_{ij}x_3 dx_3, \quad \delta N_{ij} = \int_{-h/2}^{h/2} \delta\sigma_{ij} dx_3 \tag{2.9}$$

The equilibrium equations for the perturbed state can be written in the form [16].

$$\delta M_{11,11} + 2\delta M_{12,12} + \delta M_{22,22} = h(\sigma_{11}\delta w_{,11} + \sigma_{22}\delta w_{,22}) \tag{2.10}$$

$$\delta N_{11,1}+\delta N_{12,2} = \delta X, \quad \delta N_{12,1}+\delta N_{22,2} = \delta Y \tag{2.11}$$

where δX and δY are the variation of the external surface forces, which may be non-zero even though these forces themselves are assumed to vanish in the unperturbed state.

The equation of compatibility of the variations of the strains $\delta\varepsilon_{11,22} + \delta\varepsilon_{22,11} = 2\delta e_{12,12}$, taking into account representations (2.3), is equivalent to the same equation but for variations of the strains in the middle plane

$$\delta\varepsilon^0_{11,22}+\delta\varepsilon^0_{22,11} = 2\delta\varepsilon^0_{12,12} \tag{2.12}$$

3. SOLUTION OF THE UNCOUPLED PROBLEM

In the case of the solution of the uncoupled stability problem with $\delta T = 0$, the condition $\delta q = 0$ holds. Setting $\psi(q) = 0$ in (2.8) for all points of the plate and using the first formula of (2.9), we have

$$\delta M_{ii} = -D(q)(\delta\kappa_{ii}+\mu(q)\delta\kappa_{jj}), \quad \delta M_{12} = -D(q)(1-\mu(q))\delta\kappa_{12}$$
$$D(q) = \frac{E(q)h^3}{12(1-\mu(q)^2)} \tag{3.1}$$

where $D(q)$ is the cylindrical stiffness of the plate, which takes into account the variability of the modulus of elasticity.

Substituting formulae (3.1) into the equilibrium equation (2.10), taking (2.4) into account, we obtain the stability equation for elastic plates with a coefficient depending on the phase composition parameter

$$\Delta\Delta\delta w - \frac{h}{D(q)}(\sigma_{11}\delta w_{,11}+\sigma_{22}\delta w_{,22}) = 0$$

For a plate all of whose edges are freely supported, using the representation of the deflection in double trigonometric series

$$\delta w(x_1, x_2) = w_{mn}\sin\left(\frac{m\pi}{a}x_1\right)\sin\left(\frac{n\pi}{b}x_2\right) \tag{3.2}$$

we obtain a formula

$$p_{11}\left(\frac{m}{a}\right)^2+p_{22}\left(\frac{n}{b}\right)^2 = \frac{D(q)\pi^2}{h}\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2 \tag{3.3}$$

for determining the critical loads p_{11} and p_{22}, which is identical with the well-known solution for an elastic plate but taking into account the variability of the cylindrical stiffness. Since at values of the constants of elasticity characteristic for SMAs the cylindrical stiffness decreases as q increases, that is, when there is direct transition into the martensite state, the minimum value of the critical loads (3.3) when the plate cools and with the corresponding direct transition, within the framework of the "fixed phase composition" conception, is obtained at $q = 1$. These loads are equal to the critical stresses of isothermal loss of stability in the martensite phase state. According to experimental data [5], the critical stability-loss loads in a direct martensite transition may be several times less than the same loads for isothermal stability loss in the martensite phase state. Thus, the experimental data contradict the "fixed phase composition" concept.

4. SOLUTION OF THE COUPLED PROBLEM WITHIN THE FRAMEWORK OF THE "CONTINUING LOAD" CONCEPT

Suppose the admissible perturbations include not only deflection but also external effects (load and temperature), and moreover that, by suitable choice of arbitrary small variations of these quantities δp_{11}, δp_{22} and δt, one can guarantee satisfaction of the condition $k^*\delta\sigma_i + \delta t > 0$ in the entire domain being considered. As is obvious from (2.6), an additional phase transition will take place in that case at each point of the plate. Collecting the internal moments according to formulae (2.9), using the appropriate expressions for the variations of the stresses (2.8), and noting that under these conditions $U_+ = 1$ in the case under consideration, it can be shown that

$$M_{ii} = -D\frac{\kappa_{ii}(1+\xi_{jj})+\kappa_{jj}(\mu-\xi_{ij})}{1+\xi} - \frac{E}{1-\mu^2}\frac{\psi(g_{ii}+\mu g_{jj})}{1+\xi}\int_{-h/2}^{h/2}\delta t x_3 dx_3 \tag{4.1}$$

where the expression for M_{12} does not change compared with (3.1). For brevity, the notation for the argument q in functions that depend on it are omitted in these expressions. Substituting the expression for the moments (4.1) into the equilibrium equation (2.10) we obtain an inhomogeneous partial differential equation. However, the inhomogeneous terms do not affect the critical stability-loss loads and may therefore be omitted in a stability analysis. The corresponding homogeneous equation is

$$A_{11}(q)\delta w_{,1111} + 2A_{12}(q)\delta w_{,1122} + A_{22}(q)\delta w_{,2222} - \frac{h}{D(q)}(\sigma_{11}\delta w_{,11} + \sigma_{22}\delta w_{,22}) = 0 \tag{4.2}$$

where we have introduced the notation

$$A_{ii}(q) = \frac{1+\xi_{jj}(q)}{1+\xi(q)}, \quad A_{12}(q) = 1-\mu(q) + \frac{2\mu(q)-\xi_{12}(q)-\xi_{21}(q)}{2(1+\xi(q))} \tag{4.3}$$

To sum up, for a plate freely supported at all edges, substitution of the expression (3.2) into Eq. (4.2) yields the relation

$$p_{11}\left(\frac{m}{a}\right)^2 + p_{22}\left(\frac{n}{b}\right)^2 = \frac{D(q)\pi^2}{h}\left[A_{11}(q)\left(\frac{m}{a}\right)^4 + 2A_{12}(q)\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + A_{22}(q)\left(\frac{n}{b}\right)^4\right] \tag{4.4}$$

which may be used, taking Eqs (4.3), (2.8) and (2.1) into consideration, to determine the critical loads p_{11} and p_{12}.

For a plate with freely supported transverse edges and free longitudinal edges, assuming that the variation of the deflection is independent of x_2 and that the constants of integration with respect to x_1 equal zero, one obtains the following equality instead of Eq. (4.2)

$$A_{11}(q)\delta w_{,11} - \frac{h}{D(q)}\sigma_{11}\delta w = 0 \tag{4.5}$$

Substituting into this equation the formula for the variation of the deflection

$$\delta w = w_m \sin\left(\frac{m\pi}{a}x_1\right)$$

we obtain an expression for the critical load

$$p_{11} = \left(\frac{m\pi}{a}\right)^2 \frac{D(q)A_{11}(q)}{h} \tag{4.6}$$

which in this case is a minimum when $m = 1$.

5. SOLUTION OF THE COUPLED STABILITY PROBLEM WITHIN THE FRAMEWORK OF THE "ELASTIC UNLOADING" CONCEPTION

In this case, because of unloading, the part of the plate adjoining its convex surface in the buckling process will no longer undergo a direct phase transition. Let x_3^0 be the transverse coordinate of the boundary of the zone of additional phase transition, whose value generally depends on the coordinates x_1 and x_2, which complicates an analytical solution. It can be proved, however, that if the variations of the temperature and the normal internal forces in the plate vanish.

$$\delta N_{11} = \delta N_{22} = 0 \tag{5.1}$$

$$\delta t = 0 \tag{5.2}$$

then the coordinate x_3^0 is the same for all points of the middle plane.

Indeed, the variation of the stress intensity (2.7) must vanish on the required boundary of the phase transition. Writing down this condition, taking formulae (2.8) and (5.2) into consideration and also the fact the $U_+ = 0$ on the boundary of the phase transition, we obtain

$$\begin{aligned}&\left(\sigma_{11}-\frac{\sigma_{22}}{2}\right)(\delta\varepsilon_{11}^0+\mu(q)\delta\varepsilon_{22}^0)+\left(\sigma_{22}-\frac{\sigma_{11}}{2}\right)(\delta\varepsilon_{22}^0+\mu(q)\delta\varepsilon_{11}^0) = \\ &= x_3^0\left[\left(\sigma_{11}-\frac{\sigma_{22}}{2}\right)(\delta\kappa_{11}+\mu(q)\delta\kappa_{22})+\left(\sigma_{22}-\frac{\sigma_{11}}{2}\right)(\delta\kappa_{22}+\mu(q)\kappa_{11})\right]\end{aligned} \tag{5.3}$$

Assuming that the domain of the plate cross-section $x_3 \in [-h/2, x_3^0]$ undergoes an additional phase transition, while the rest of the cross-section does not, one obtains the following expressions for the variations of the internal forces (2.9) δN_{11}, δN_{22} and δN_{12}

$$\begin{aligned}\delta N_{ii} &= \frac{E}{1-\mu^2}\left\{\frac{\delta\varepsilon_{ii}^0+\mu\delta\varepsilon_{jj}^0}{1+\xi}\left[h+\xi\left(\frac{h}{2}-x_3^0\right)\right]+\frac{\delta\varepsilon_{ii}^0\xi_{jj}-\delta\varepsilon_{jj}^0\xi_{ij}}{1+\xi}\left(\frac{h}{2}+x_3^0\right)-\right.\\ &\left.-\frac{1}{2}\left(\frac{h^2}{4}-(x_3^0)^2\right)\left[\frac{(\delta\kappa_{ii}+\mu\delta\kappa_{jj})\xi}{1+\xi}-\frac{\delta\kappa_{ii}\xi_{jj}-\delta\kappa_{jj}\xi_{ij}}{1+\xi}\right]\right\}\end{aligned} \tag{5.4}$$

$$\delta N_{12} = 2Gh\delta\varepsilon_{12}^0$$

Multiplying equation (5.4) for δN_{11} by $(\sigma_{11} - \sigma_{22}/2)$ and the equation for δN_{22} by $(\sigma_{22} - \sigma_{11}/2)$ and adding the results together, one arrives, via Eq. (5.3), at a relation which, together with conditions (5.1), enables one to find the required value of the coordinate x_3^0

$$x_3^0 = \frac{h}{\xi}\left(1+\frac{\xi}{2}-\sqrt{1+\xi}\right) \tag{5.5}$$

By (5.5), the quantity x_3^0 is indeed independent of x_1 and x_2 on the assumption that (5.1) and (5.2) are true.

Substituting expressions (2.8) into the first formula of (2.9) we obtain expressions for the variations of the internal moments δM_{11} and δM_{22}, which contain terms with the variations both of the curvatures $\delta\kappa_{11}$ and $\delta\kappa_{22}$ and of the strains of the middle plane $\delta\varepsilon_{11}^0$ and $\delta\varepsilon_{22}^0$. Eliminating the latter using (5.1) and (5.4) and also using formula (5.5), one arrives, after some relatively simple but very cumbersome algebra, at the following expressions

$$\delta M_{ii} = -D(q)\left\{\left[\frac{\xi_{jj}}{\xi}+\left(1-\frac{\xi_{jj}}{\xi}\right)\frac{8x_3^0}{h\xi}\right]\delta\kappa_{ii}+\left[-\frac{\xi_{ij}}{\xi}+\left(\mu+\frac{\xi_{ij}}{\xi}\right)\frac{8x_3^0}{h\xi}\right]\delta\kappa_{jj}\right\} \tag{5.6}$$

The formula obtained for δM_{12} is identical with (3.1). Substituting the latter, together with (5.6), into the equilibrium equation (2.1) we obtain the required differential equation, which is identical in form with (4.2) but has different coefficients:

$$\begin{aligned}A_{ii}(q) &= \frac{\xi_{jj}(q)}{\xi(q)}+\left(1-\frac{\xi_{jj}(q)}{\xi(q)}\right)\frac{8x_3^0}{h\xi(q)}\\ A_{12}(q) &= 1-\mu(q)-\frac{\xi_{12}(q)+\xi_{21}(q)}{2\xi(q)}+\left(\mu(q)+\frac{\xi_{12}(q)+\xi_{21}(q)}{2\xi(q)}\right)\frac{8x_3^0}{h\xi(q)}\end{aligned} \tag{5.7}$$

For a plate freely supported along all its edges, given representation (3.2) of the variation of the deflection, this equation implies formula (4.4), but with coefficients computed from formulae (5.7), and this relation may be used, with due consideration of formulae (2.8) and (2.1), to determine the critical load parameters p_{11} and p_{22}.

It should be noted that assumptions (5.1), subject to which this solution has been found, do not contradict the boundary conditions for the forces at the ends of the plate if the variations of the normal forces at the ends vanish. In this case, therefore, what we have is a loss of stability at fixed normal external forces.

It is also particularly noteworthy that in this solution the variation of the shearing force δN_{12} does not vanish. Its value may be determined by using the first two formulae of (5.4), on the assumption that condition (5.1) is satisfied, to express the variations $\delta\varepsilon_{11}^0$ and $\delta\varepsilon_{22}^0$ in terms of the curvatures $\delta\kappa_{11}$ and $\delta\kappa_{22}$ and the last equation of (5.4) to express the variation of the shear strain $\delta\varepsilon_{12}^0$ in terms of δN_{12}, subsequently substituting these values into the compatibility equation (2.12) for the strains. The result is a formula for the second mixed derivative for a freely supported plate, taking (3.2) into account:

$$\delta N_{12,12}(x_1, x_2) = \delta N_{12mn}\frac{mn\pi^2}{ab}\sin\left(\frac{m\pi}{a}x_1\right)\sin\left(\frac{n\pi}{b}x_2\right)$$

in which we have introduced the notation

$$\delta N_{12mn} = \frac{G}{1-\mu^2}\frac{hx_3^0 a\, b}{\xi\; mn}\pi^2 w_{mn}\left[(\mu\xi_{22}+\xi_{21})\frac{m^4}{a^4}+(1-\mu^2)\xi\frac{m^2n^2}{a^2b^2}+(\mu\xi_{11}+\xi_{12})\frac{n^4}{b^4}\right]$$

Consequently, the variation of the shearing force is given by the formula

$$\delta N_{12}(x_1, x_2) = \delta N_{12mn}\cos\left(\frac{m\pi}{a}x_1\right)\cos\left(\frac{n\pi}{b}x_2\right) \tag{5.8}$$

using which, one infers from the equilibrium equations (2.11) and from conditions (5.1) that the variations of the surface forces δX and δY must be found from the formulae

$$\begin{aligned}\delta X(x_1, x_2) &= -\delta N_{12mn}\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x_1\right)\sin\left(\frac{n\pi}{b}x_2\right)\\ \delta Y(x_1, x_2) &= -\delta N_{12mn}\frac{m\pi}{a}\sin\left(\frac{m\pi}{a}x_1\right)\cos\left(\frac{n\pi}{b}x_2\right)\end{aligned} \tag{5.9}$$

In addition, the variations of the shearing forces at opposite edges of the plate must not vanish:

$$\delta p_{12}(0, x_2) = \frac{1}{h}\delta N_{12mn}(0, x_2), \quad \delta p_{12}(a, x_2) = \frac{1}{h}\delta N_{12mn}(a, x_2) \tag{5.10}$$

$$\delta p_{21}(x_1, 0) = \frac{1}{h}\delta N_{12mn}(x_1, 0), \quad \delta p_{21}(x_1, b) = \frac{1}{h}\delta N_{12mn}(x_1, b) \tag{5.11}$$

As is obvious, all the quantities determined by formulae (5.8)–(5.1) have been found, apart from a small factor w_{mn}, from expression (3.2) for the variation of the deflection. Thus, the perturbations of the external shearing loads required to implement this solution are small.

It should be noted that, for a plate with free longitudinal edges, formulae (5.5) and (5.7) hold provided that the variations of the external longitudinal loads vanish without any further assumptions. The equilibrium equations are satisfied at zero values of the variations of the shearing forces.

6. LOSS OF STABILITY OF A BIAXIALLY LOADED SQUARE PLATE

As an example illustrating the different approaches considered in this paper, let us investigate a square plate of SMA loaded at two pairs of opposite edges by constant surface loads p_{11} and p_{22} and under conditions of direct martensite transition. The dimensionless parameters of the material (corresponding to titanium nickelide with equal atomic content of nickel and titanium) are

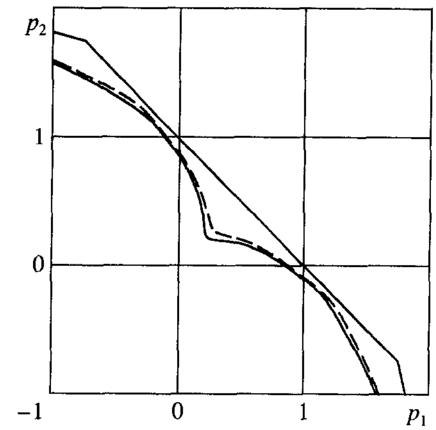

Fig. 1

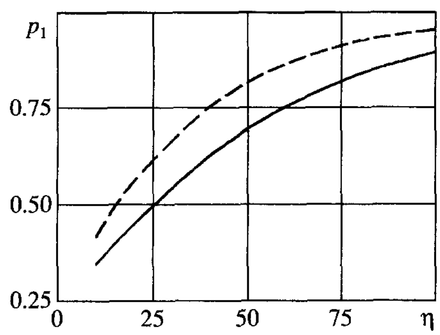

Fig. 2

$$\frac{a}{b} = 1, \quad \frac{h}{b} = \frac{1}{20}, \quad \frac{E_1}{E_2} = \frac{1}{3}, \quad \mu_1 = 0.48, \quad \mu_2 = 0.33, \quad k^*E_2 = 480,$$

$$\frac{\sigma_{(1)}}{E_2} = 0.049, \quad a_0 = 0.718$$

Figure 1 shows the boundaries of the stability domains of such a plate. The axes measure the dimensionless critical loads $p_1 = p_{11}/p^*$ and $p_2 = p_{22}/p^*$, where $p^* = 4D(1)\pi^2/(hb^2)$ is the critical load of isothermal loss of stability of such a plate under uniaxial compression in the martensite state. The solution was obtained with the phase transition diagram approximated by formula (1.3). The polygonal line corresponds to the boundary of the stability domain obtained by analysis of Eq. (3.3), corresponding to the solution of the unconnected problem. The solid curve corresponds to the solution of the connected problem assuming the "continuing loading" hypothesis. The dashed curve corresponds to the solution of the connected problem, assuming the "elastic unloading" conception.

As can be seen from Fig. 1, the least critical loads are obtained under the "continuing phase transition" and "continuing loading" hypotheses. This seems to be quite justified, since in this case the plate, owing to the additional phase transition at all its points, is most inclined to develop phase strains, i.e., it is most pliable. It should be noted that the boundaries of the stability domains for the last two approaches considered are not convex.

According to Fig. 1, the critical loads determined using the different hypotheses for a square plate show the greatest discrepancy in the case of biaxial loading by identical loads: $p_{11} = p_{22} = p$. In Fig. 2 the dimensionless critical values of $p_1 = p/p^{**}$ are plotted versus the quantity $\eta = a/h$. Here the critical load is considered relative to the maximum load p^{**} of isothermal stability loss in the martensite state under equi-biaxial compression or, what is the same, relative to the minimum critical load obtained by

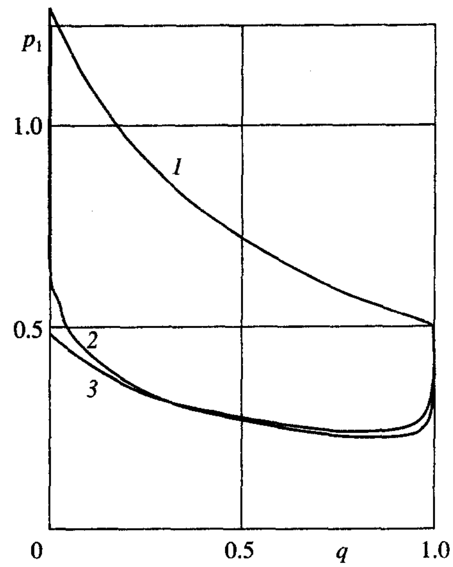

Fig. 3

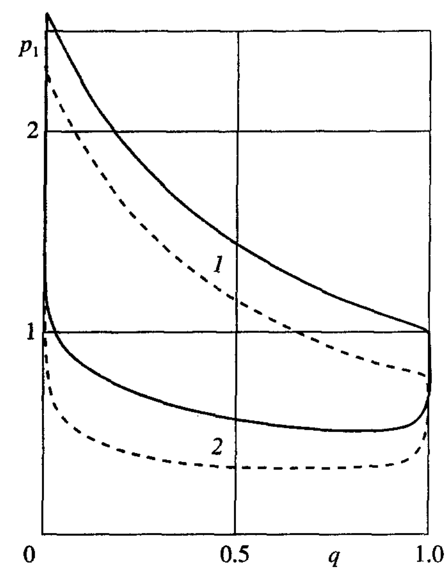

Fig. 4

solving the problem in the unconnected formulation, which thus corresponds to the upper horizontal line $p_1 = 1$. The solution of the connected problem, assuming the "elastic unloading" hypothesis, is represented by the dashed curve, and the solution assuming the "continuing unloading" conception is represented by the solid curve. As is clearly seen, the difference between the solutions is small for sufficiently thin plates; it increases as the relative thickness increases (as η decreases). The differences between the critical stability-loss loads in the direct transformation and the isothermal stability-loss loads in the least rigid martensite state, for fairly thick plates, may amount to several factors, in agreement with the experimental data [5].

It was never possible in the experiments in [5] to obtain a loss of stability in a direct transition at the last point of the interval of this transition, that is, at $q = 1$. The deviation from the rectilinear shape always took place at certain intermediate temperatures, considerably exceeding the temperature at the end of the direct transition reaction. In Fig. 3 the minimum (with respect to m and n) dimensionless critical load parameter $p_1 = p_{11}/p^*$ is plotted versus the relative volume q of the martensite phase in the problem of stability loss for direct transition in biaxial compression by equal loads. Curve 1 corresponds to the solution of the unconnected problem, curve 2 corresponds to the solution of the connected problem, assuming the "continuing loading" conception and with the phase transition diagram approximated by the second formula of (1.3) and curve 3 represents the solution of the same problem, but with the phase transition diagram approximated by the first formula of (1.3). As can be clearly seen, in the solution of the problem in the coupled statement, unlike the results for the uncoupled problem, the minimum of the critical load never corresponds to the value $q = 1$, that is, loss of stability occurs at an intermediate point of the direct transition, rather than at the ends; interval this also does not contradict the experimental data. The minima on the curves plotted in Fig. 3 and constructed for different approximations of the phase diagram are close together, though the curves themselves are far apart at $q = 0$. This difference is due to the fact that, by the second formula of (1.3), one has $dq/dT = 0$ at $T = M_1$, that is, at $q = 0$, whereas according to the first formula of (1.3) the derivative will never vanish at the initial point of the interval of direct transition temperatures, but will have the maximum absolute value at the point.

Figure 4 compares the solutions of the stability problem for a plate with free longitudinal sides, but freely supported transverse sides (4.6), and for a rod freely supported at both ends, assuming that the cross-sections have identical moments of inertia and that the plate and rod are of the same length. The solution of the stability problem for a rod is taken from [6]. In both cases the "continuing loading" conception is assumed and the second formula of (1.3) is used for the phase diagram. Critical loads, relative to the maximum load of isothermal stability loss for a strip in the martensite state, are plotted along the ordinate axis. The solid curves in Fig. 4 correspond to the plate and the dashed curves to the rod. Curve 1 are constructed for the solution of the uncoupled problem and curves 2 for the solution of the coupled problem. It is obvious that the relative difference between the critical stability loss loads for a plate and a rod, obtained when the coupled problem is solved, is markedly greater than that obtained for the uncoupled problem.

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